

# The two-dimensional moment problem in a strip.

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## 1 Introduction.

In this paper we consider the following problem: to find a non-negative Borel measure  $\mu$  in a strip

$$\Pi = \Pi(R) = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \leq R\}, \quad R > 0,$$

such that

$$\int_{\Pi} x_1^m x_2^n d\mu = s_{m,n}, \quad m, n \in \mathbb{Z}_+, \quad (1)$$

where  $\{s_{m,n}\}_{m,n \in \mathbb{Z}_+}$  is a prescribed sequence of complex numbers. This problem is said to be **the two-dimensional moment problem in a strip**.

The two-dimensional moment problem and the complex moment problem have an extensive literature, see books [1], [2], surveys [3],[4] and [5]. However, to the best of our knowledge, the two-dimensional moment problem in a strip was not solved.

Firstly, we obtain a solvability criterion for the two-dimensional moment problem in a strip. We describe canonical solutions of this moment problem (see the definition below). Secondly, we parameterize all solutions of the moment problem. In a consequence, we derive conditions of the solvability and describe all solutions of the complex moment problem with the support in a strip. We shall use an abstract operator approach [6] and results of Godič, Lucenko and Shtraus [7],[8, Theorem 1],[9].

**Notations.** As usual, we denote by  $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$  the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. For a subset  $S$  of the complex plane we denote by  $\mathfrak{B}(S)$  the set of all Borel subsets of  $S$ . Everywhere in this paper, all Hilbert spaces are assumed to be separable. By  $(\cdot, \cdot)_H$  and  $\|\cdot\|_H$  we denote the scalar product and the norm in a Hilbert space  $H$ , respectively. The indices may be omitted in obvious cases. For a set  $M$  in  $H$ , by  $\overline{M}$  we mean the closure of  $M$  in the norm  $\|\cdot\|_H$ . For  $\{x_{m,n}\}_{m,n \in \mathbb{Z}_+}$ ,  $x_{m,n} \in H$ , we write  $\text{Lin}\{x_{m,n}\}_{m,n \in \mathbb{Z}_+}$  for the span of vectors  $\{x_{m,n}\}_{m,n \in \mathbb{Z}_+}$  and  $\text{span}\{x_{m,n}\}_{m,n \in \mathbb{Z}_+} = \overline{\text{Lin}\{x_{m,n}\}_{m,n \in \mathbb{Z}_+}}$ . The identity operator in  $H$  is denoted by  $E_H$ . For an arbitrary linear operator  $A$  in  $H$ , the operators  $A^*, \overline{A}, A^{-1}$  mean its adjoint operator, its closure and its inverse (if they exist). By  $D(A)$  and  $R(A)$  we mean the domain and the range of the operator  $A$ . By  $\sigma(A)$ ,  $\rho(A)$  we denote the spectrum of  $A$ .

and the resolvent set of  $A$ , respectively. We denote by  $R_z(A)$  the resolvent function of  $A$ ,  $z \in \rho(A)$ ;  $\Delta_A(z) = (A - zE_H)D(A)$ ,  $z \in \mathbb{C}$ . The norm of a bounded operator  $A$  is denoted by  $\|A\|$ . By  $P_{H_1}^H = P_{H_1}$  we mean the operator of orthogonal projection in  $H$  on a subspace  $H_1$  in  $H$ . By  $\mathbf{B}(H)$  we denote the set of all bounded operators in  $H$ .

## 2 Solvability of the two-dimensional moment problem in a strip.

Let the two-dimensional moment problem in a strip (1) be given. Suppose that the moment problem has a solution  $\mu$ . Choose an arbitrary polynomial  $p(x_1, x_2)$  of the following form:

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} x_1^m x_2^n, \quad \alpha_{m,n} \in \mathbb{C}, \quad (2)$$

where all but finite number of coefficients  $\alpha_{m,n}$  are zeros. We may write

$$\begin{aligned} 0 &\leq \int_{\Pi} |p(x_1, x_2)|^2 d\mu = \int_{\Pi} \sum_{m,n=0}^{\infty} \alpha_{m,n} x_1^m x_2^n \overline{\sum_{k,l=0}^{\infty} \alpha_{k,l} x_1^k x_2^l} d\mu \\ &= \sum_{m,n,k,l} \alpha_{m,n} \overline{\alpha_{k,l}} \int_{\Pi} x_1^{m+k} x_2^{n+l} d\mu = \sum_{m,n,k,l} \alpha_{m,n} \overline{\alpha_{k,l}} s_{m+k,n+l}. \end{aligned}$$

Thus, for arbitrary complex numbers  $\alpha_{m,n}$  (where all but finite numbers are zeros) we have

$$\sum_{m,n,k,l=0}^{\infty} \alpha_{m,n} \overline{\alpha_{k,l}} s_{m+k,n+l} \geq 0. \quad (3)$$

Since

$$\int_{\Pi} |x_2 p(x_1, x_2)|^2 d\mu \leq R^2 \int_{\Pi} |p(x_1, x_2)|^2 d\mu,$$

in a similar manner we get

$$\sum_{m,n,k,l=0}^{\infty} \alpha_{m,n} \overline{\alpha_{k,l}} (R^2 s_{m+k,n+l} - s_{m+k,n+l+2}) \geq 0, \quad (4)$$

for arbitrary complex numbers  $\alpha_{m,n}$  (where all but finite numbers are zeros).

On the other hand, suppose that the moment problem (1) is given and conditions (3) and (4) hold. Let us show that the moment problem has a solution. Set

$$K((m, n), (k, l)) = s_{m+k, n+l}, \quad m, n, k, l \in \mathbb{Z}_+. \quad (5)$$

Then relations (3) may be written as

$$\sum_{m, n, k, l=0}^{\infty} \alpha_{m, n} \overline{\alpha_{k, l}} K((m, n), (k, l)) \geq 0, \quad (6)$$

for arbitrary complex numbers  $\alpha_{m, n}$  (where all but finite numbers are zeros). In this case  $K$  is said to be a *positive definite kernel on  $\mathbb{Z}_+ \times \mathbb{Z}_+$* .

We shall use the following important fact (e.g. [10, pp.361-363]).

**Theorem 2.1** *Let  $K$  be a positive definite kernel on  $\mathbb{Z}_+ \times \mathbb{Z}_+$ . Then there exist a separable Hilbert space  $H$  with a scalar product  $(\cdot, \cdot)$  and a sequence  $\{x_{m, n}\}_{m, n \in \mathbb{Z}_+}$  in  $H$ , such that*

$$K((m, n), (k, l)) = (x_{m, n}, x_{k, l}), \quad m, n, k, l \in \mathbb{Z}_+, \quad (7)$$

and  $\text{span}\{x_{m, n}\}_{m, n \in \mathbb{Z}_+} = H$ .

**Proof.** Choose an arbitrary infinite-dimensional linear vector space  $V$  (for instance, we may choose the space of all complex sequences  $(u_n)_{n \in \mathbb{N}}$ ,  $u_n \in \mathbb{C}$ ). Let  $X = \{x_{m, n}\}_{m, n \in \mathbb{Z}_+}$  be an arbitrary infinite sequence of linear independent elements in  $V$  which is indexed by elements of  $\mathbb{Z}_+ \times \mathbb{Z}_+$ . Set  $L_X = \text{Lin}\{x_{m, n}\}_{m, n \in \mathbb{Z}_+}$ . Introduce the following functional:

$$[x, y] = \sum_{m, n, k, l=0}^{\infty} K((m, n), (k, l)) a_{m, n} \overline{b_{k, l}}, \quad (8)$$

for  $x, y \in L_X$ ,

$$x = \sum_{m, n=0}^{\infty} a_{m, n} x_{m, n}, \quad y = \sum_{k, l=0}^{\infty} b_{k, l} x_{k, l}, \quad a_{m, n}, b_{k, l} \in \mathbb{C}.$$

Here all but finite number of indices  $a_{m, n}, b_{k, l}$  are zeros.

The set  $L_X$  with  $[\cdot, \cdot]$  will be a pre-Hilbert space. Factorizing and making the completion we obtain the desired space  $H$  ([11, p. 10-11]).  $\square$

By Theorem 2.1 we obtain a Hilbert space  $H$  and a sequence  $\{x_{m,n}\}_{m,n \in \mathbb{Z}_+}$ ,  $x_{m,n} \in H$ , such that

$$(x_{m,n}, x_{k,l})_H = K((m,n), (k,l)), \quad m, n, k, l \in \mathbb{Z}_+. \quad (9)$$

Set  $L = \text{Lin}\{x_{m,n}\}_{m,n \in \mathbb{Z}_+}$ . Introduce the following operators

$$A_0 x = \sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} x_{m+1,n}, \quad (10)$$

$$B_0 x = \sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} x_{m,n+1}, \quad (11)$$

where

$$x = \sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} x_{m,n} \in L. \quad (12)$$

Let us check that these definitions are correct. Indeed, suppose that the element  $x$  in (12) has another representation:

$$x = \sum_{k,l \in \mathbb{Z}_+} \beta_{k,l} x_{k,l}. \quad (13)$$

We may write

$$\begin{aligned} \left( \sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} x_{m+1,n}, x_{a,b} \right) &= \sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} K((m+1,n), (a,b)) \\ &= \sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} s_{m+1+a,n+b} = \sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} K((m,n), (a+1,b)) \\ &= \left( \sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} x_{m,n}, x_{a+1,b} \right) = (x, x_{a+1,b}), \end{aligned}$$

for arbitrary  $a, b \in \mathbb{Z}_+$ . In the same manner we get

$$\left( \sum_{k,l \in \mathbb{Z}_+} \beta_{k,l} x_{k+1,l}, x_{a,b} \right) = (x, x_{a+1,b}).$$

Since  $\text{span}\{x_{a,b}\}_{a,b \in \mathbb{Z}_+} = H$ , we get

$$\sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} x_{m+1,n} = \sum_{k,l \in \mathbb{Z}_+} \beta_{k,l} x_{k+1,l}.$$

Therefore the operator  $A_0$  is defined correctly. For  $B_0$  considerations are similar. It is not hard to see that operators  $A_0$  and  $B_0$  are symmetric. Moreover, condition (4) implies that the operator  $B_0$  is bounded. Set

$$A = \overline{A_0}, \quad B = \overline{B_0}. \quad (14)$$

Observe that  $B_0$  is a bounded self-adjoint operator in  $H$ . Since  $A_0$  and  $B_0$  commute, we easily get

$$ABx = BAx, \quad x \in D(A). \quad (15)$$

We shall also need the following operator:

$$J_0 x = \sum_{m,n \in \mathbb{Z}_+} \overline{\alpha_{m,n}} x_{m,n}, \quad (16)$$

where

$$x = \sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} x_{m,n} \in L. \quad (17)$$

Let us check that this definition is correct. Consider another representation for  $x$  as in (13). Then

$$\begin{aligned} & \left\| \sum_{m,n \in \mathbb{Z}_+} (\overline{\alpha_{m,n}} - \overline{\beta_{m,n}}) x_{m,n} \right\|^2 \\ &= \left( \sum_{m,n \in \mathbb{Z}_+} \overline{(\alpha_{m,n} - \beta_{m,n})} x_{m,n}, \sum_{k,l \in \mathbb{Z}_+} (\alpha_{k,l} - \beta_{k,l}) x_{k,l} \right) \\ &= \sum_{m,n,k,l \in \mathbb{Z}_+} \overline{(\alpha_{m,n} - \beta_{m,n})} (\alpha_{k,l} - \beta_{k,l}) K((m,n), (k,l)) \\ &= \sum_{m,n,k,l \in \mathbb{Z}_+} \overline{(\alpha_{m,n} - \beta_{m,n})} (\alpha_{k,l} - \beta_{k,l}) K((k,l), (m,n)) \\ &= \left( \sum_{m,n \in \mathbb{Z}_+} (\alpha_{k,l} - \beta_{k,l}) x_{k,l}, \sum_{m,n \in \mathbb{Z}_+} (\alpha_{m,n} - \beta_{m,n}) x_{m,n} \right) = 0. \end{aligned}$$

Therefore the definition of  $J_0$  is correct. For an arbitrary  $y = \sum_{a,b \in \mathbb{Z}_+} \gamma_{a,b} x_{a,b} \in L$  we may write

$$(J_0 x, J_0 y) = \sum_{m,n,a,b} \overline{\alpha_{m,n}} \gamma_{a,b} (x_{m,n}, x_{a,b}) = \sum_{m,n,a,b} \overline{\alpha_{m,n}} \gamma_{a,b} K((m,n), (a,b))$$

$$= \sum_{m,n,a,b} \overline{\alpha_{m,n}} \gamma_{a,b} K((a,b), (m,n)) = \sum_{m,n,a,b} \overline{\alpha_{m,n}} \gamma_{a,b} (x_{a,b}, x_{m,n}) = (y, x).$$

In particular, this implies that  $J_0$  is bounded. By continuity we extend  $J_0$  to a bounded antilinear operator  $J$  such that

$$(Jx, Jy) = (y, x), \quad x, y \in H.$$

Moreover, we get  $J^2 = E_H$ . Consequently,  $J$  is a conjugation in  $H$  ([12]). Notice that  $J_0$  commutes with  $A_0$  and  $B_0$ . It is easy to check that

$$AJx = JAx, \quad x \in D(A), \quad (18)$$

and by continuity we have

$$BJx = JBx, \quad x \in H. \quad (19)$$

Consider the Cayley transformations of the operators A,B:

$$V_A := (A + iE_H)(A - iE_H)^{-1} = E + 2i(A - iE_H)^{-1}, \quad (20)$$

$$U_B := (B + iE_H)(B - iE_H)^{-1} = E + 2i(B - iE_H)^{-1}. \quad (21)$$

Set

$$H_1 := \Delta_A(i), \quad H_2 := H \ominus H_1, \quad H_3 := \Delta_A(-i), \quad H_4 := H \ominus H_3. \quad (22)$$

**Proposition 2.1** *The operator  $U_B$  reduces the subspaces  $H_j$ ,  $1 \leq j \leq 4$ :*

$$U_B H_j = H_j, \quad 1 \leq j \leq 4. \quad (23)$$

Moreover, the following equality holds:

$$U_B V_A x = V_A U_B x, \quad x \in H_1. \quad (24)$$

**Proof.** Choose an arbitrary  $x \in \Delta_A(z)$ ,  $x = (A - zE_H)f_A$ ,  $f_A \in D(A)$ ,  $z \in \mathbb{C} \setminus \mathbb{R}$ . By (15) we get

$$Bx = BAf_A - zBf_A = ABf_A - zBf_A = (A - zE_H)Bf_A \in \Delta_A(z).$$

In particular, we have

$$BH_1 \subseteq H_1, \quad BH_3 \subseteq H_3. \quad (25)$$

Since  $B$  is bounded and self-adjoint, we get

$$BH_2 \subseteq H_2, \quad BH_4 \subseteq H_4. \quad (26)$$

This implies equalities

$$(B - iE_H)H_j = H_j, \quad 1 \leq j \leq 4; \quad (27)$$

$$(B - iE_H)^{-1}H_j = H_j, \quad 1 \leq j \leq 4. \quad (28)$$

Then  $U_B H_j \subseteq H_j$ ,  $1 \leq j \leq 4$ , and relation (23) follows.

Since

$$(A - iE_H)Bx = B(A - iE_H)x, \quad x \in D(A),$$

we get

$$BV_A y = V_A B y, \quad y \in H_1. \quad (29)$$

Then

$$(B - iE_H)V_A y = V_A(B - iE_H)y, \quad y \in H_1,$$

and, hence, we easily get relation (24).  $\square$

We shall construct a unitary operator  $U$  in  $H$ ,  $U \supset V_A$ , which commutes with  $U_B$ . Choose an arbitrary  $x \in H$ ,  $x = x_{H_1} + x_{H_2}$ ,  $x_{H_1} \in H_1$ ,  $x_{H_2} \in H_2$ . If  $U$  is an operator of the required type then (we use Proposition 2.1):

$$U_B U x = U_B V_A x_{H_1} + U_B U x_{H_2} = V_A U_B x_{H_1} + U_B U x_{H_2},$$

$$U U_B x = U U_B x_{H_1} + U U_B x_{H_2} = V_A U_B x_{H_1} + U U_B x_{H_2}.$$

Thus, we have to find an isometric operator  $U_{2,4}$  which maps  $H_2$  onto  $H_4$ , and commutes with  $U_B$ :

$$U_B U_{2,4} x = U_{2,4} U_B x, \quad x \in H_2. \quad (30)$$

Moreover, all operators  $U$  of the required type have the following form:

$$U = V_A \oplus U_{2,4}, \quad (31)$$

where  $U_{2,4}$  is an isometric operator which maps  $H_2$  onto  $H_4$ , and commutes with  $U_B$ .

Denote the operator  $U_B$  restricted to  $H_i$  by  $U_{B;H_i}$ ,  $1 \leq i \leq 4$ . Notice that

$$A^* J x = J A^* x, \quad x \in D(A^*). \quad (32)$$

Indeed, for arbitrary  $f_A \in D(A)$  and  $g_{A^*} \in D(A^*)$  we may write

$$\overline{(A f_A, J g_{A^*})} = (J A f_A, g_{A^*}) = (A J f_A, g_{A^*}) = (J f_A, A^* g_{A^*})$$

$$= \overline{(f_A, JA^*g_{A^*})},$$

and (32) follows.

Choose an arbitrary  $x \in H_2$ . We have

$$A^*x = -ix,$$

and therefore

$$A^*Jx = JA^*x = ix.$$

Thus, we have

$$JH_2 \subseteq H_4.$$

In a similar manner we get

$$JH_4 \subseteq H_2,$$

and therefore

$$JH_2 = H_4, \quad JH_4 = H_2. \quad (33)$$

By the Godič-Lucenko Theorem ([7],[8, Theorem 1]) we have a representation:

$$U_{B;H_2} = KL, \quad (34)$$

where  $K$  and  $L$  are some conjugations in  $H_2$ . We set

$$U_{2,4} := JK. \quad (35)$$

From (33) it follows that  $U_{2,4}$  maps isometrically  $H_2$  onto  $H_4$ . Observe that

$$U_{2,4}^{-1} := KJ. \quad (36)$$

Notice that

$$JU_BJ = U_B^{-1}. \quad (37)$$

Indeed, by virtue of [13, Proposition 2.10] we can write

$$\begin{aligned} JU_BJ &= E - 2iJ(B - iE_H)^{-1}J = E - 2i(J(B - iE_H)J)^{-1} \\ &= E - 2i(B + iE_H)^{-1} = E - 2iR_B(-i) = U_B^* = U_B^{-1}. \end{aligned}$$

By (37) we get

$$\begin{aligned} U_{2,4}U_{B;H_2}U_{2,4}^{-1}x &= JK K L K J x = J L K J x = JU_{B;H_2}^{-1}Jx \\ &= JU_B^{-1}Jx = U_Bx = U_{B;H_4}x, \quad x \in H_4. \end{aligned}$$



Therefore relation (30) is true. We define an operator  $U$  by (31) and set

$$A_U := i(U + E_H)(U - E_H)^{-1} = iE_H + 2i(U - E_H)^{-1}. \quad (38)$$

The inverse Cayley transformation  $A_U$  is correctly defined since 1 is not in the point spectrum of  $U$ . Indeed,  $V_A$  is the Cayley transformation of a symmetric operator while eigen subspaces  $H_2$  and  $H_4$  have the zero intersection. Let

$$A_U = \int_{\mathbb{R}} x_1 dE(x_1), \quad B = \int_{[-R, R]} x_2 dF(x_2), \quad (39)$$

where  $E$  and  $F$  are the spectral measures of  $A_U$  and  $B$ , respectively. These measures are defined on  $\mathfrak{B}(\mathbb{R})$  ([14]). Since  $U$  and  $U_B$  commute, we get that  $E$  and  $F$  commute, as well. By the induction argument we get

$$x_{m,n} = A^m x_{0,n}, \quad m, n \in \mathbb{Z}_+,$$

and

$$x_{0,n} = B^n x_{0,0}, \quad n \in \mathbb{Z}_+.$$

Therefore we obtain

$$x_{m,n} = A^m B^n x_{0,0}, \quad m, n \in \mathbb{Z}_+. \quad (40)$$

We may write

$$x_{m,n} = \int_{\mathbb{R}} x_1^m dE(x_1) \int_{[-R, R]} x_2^n dF(x_2) x_{0,0} = \int_{\Pi} x_1^m x_2^n d(E \times F)(x_1, x_2) x_{0,0},$$

where  $E \times F$  is the product spectral measure on  $\mathfrak{B}(\Pi)$ . Then

$$s_{m,n} = (x_{m,n}, x_{0,0})_H = \int_{\Pi} x_1^m x_2^n d((E \times F)x_{0,0}, x_{0,0})_H, \quad m, n \in \mathbb{Z}_+. \quad (41)$$

The measure  $\mu := ((E \times F)x_{0,0}, x_{0,0})_H$  is a non-negative Borel measure on  $\Pi$  and relation (41) shows that  $\mu$  is a solution of the moment problem (1).

**Theorem 2.2** *Let the moment problem (1) be given. This problem has a solution if and only if conditions (3), (4) hold for arbitrary complex numbers  $\alpha_{m,n}$  such that all but finite numbers are zeros.*

### 3 A parameterization of all solutions of the two-dimensional moment problem in a strip.

Let the moment problem (1) be given. Define a Hilbert space  $H$  and operators  $A, B, J$  as in the previous Section. Let  $\tilde{A} \supseteq A$  be a self-adjoint extension of  $A$  in a Hilbert space  $\tilde{H} \supseteq H$  and  $E_{\tilde{A}}$  be the spectral measure of  $\tilde{A}$ . Recall that the function

$$\mathbf{R}_z(A) := P_H^{\tilde{H}} R_z(\tilde{A}), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (42)$$

is said to be a *generalized resolvent* of  $A$ . The function

$$\mathbf{E}_A(\delta) := P_H^{\tilde{H}} E_{\tilde{A}}(\delta), \quad \delta \in \mathfrak{B}(\mathbb{R}), \quad (43)$$

is said to be a *spectral measure* of  $A$ . There exists a one-to-one correspondence between generalized resolvents and spectral measures according to the following relation [10]:

$$(\mathbf{R}_z(A)x, y)_H = \int_{\mathbb{R}} \frac{1}{t - z} d(\mathbf{E}_A x, y)_H, \quad x, y \in H. \quad (44)$$

**Theorem 3.1** *Let the moment problem (1) be given and conditions (3), (4) hold. Consider a Hilbert space  $H$  and a sequence  $\{x_{m,n}\}_{m,n \in \mathbb{Z}_+}$ ,  $x_{m,n} \in H$ , such that relation (9) holds where  $K$  is defined by (4). Consider operators  $A_0, B_0, A, B$  defined by (10), (11) and (14). Let  $\mu$  be an arbitrary solution of the moment problem. Then it has the following form:*

$$\mu(\delta) = ((\mathbf{E} \times F)(\delta)x_{0,0}, x_{0,0})_H, \quad \delta \in \mathfrak{B}(\Pi), \quad (45)$$

where  $F$  is the spectral measure of  $B$ ,  $\mathbf{E}$  is a spectral measure of  $A$  which commutes with  $F$ . By  $((\mathbf{E} \times F)(\delta)x_{0,0}, x_{0,0})_H$  we mean the non-negative Borel measure on  $\mathbb{R}$  which is obtained by the Lebesgue continuation procedure from the following non-negative measure on rectangles

$$((\mathbf{E} \times F)(I_{x_1} \times I_{x_2})x_{0,0}, x_{0,0})_H := (\mathbf{E}(I_{x_1})F(I_{x_2})x_{0,0}, x_{0,0})_H, \quad (46)$$

where  $I_{x_1} \subset \mathbb{R}$ ,  $I_{x_2} \subseteq [-R, R]$  are arbitrary intervals.

On the other hand, for an arbitrary spectral measure  $\mathbf{E}$  of  $A$  which commutes with the spectral measure  $F$  of  $B$ , by relation (45) it corresponds a solution of the moment problem (1).

Moreover, the correspondence between the spectral measures of  $A$  which commute with the spectral measure of  $B$  and solutions of the moment problem is bijective.

**Remark.** It is straightforward to check that the measure in (46) is non-negative and additive. Moreover, the standard arguments [15, Chapter 5, Theorem 2, p. 254-255] imply that the measure in (46) is  $\sigma$ -additive. Consequently, it has the (unique) Lebesgue continuation to a (finite) non-negative Borel measure on  $\Pi$ .

**Proof.** Consider a Hilbert space  $H$  and operators  $A, B$  as in the statement of the Theorem. Let  $F$  be the spectral measure of  $B$ . Let  $\mu$  be an arbitrary solution of the moment problem (1). Consider the space  $L_\mu^2$  of complex functions on  $\Pi$  which are square integrable with respect to the measure  $\mu$ . The scalar product and the norm are given by

$$(f, g)_\mu = \int_\Pi f(x_1, x_2) \overline{g(x_1, x_2)} d\mu, \quad \|f\|_\mu = ((f, f)_\mu)^{\frac{1}{2}}, \quad f, g \in L_\mu^2.$$

Consider the following operators:

$$A_\mu f(x_1, x_2) = x_1 f(x_1, x_2), \quad D(A_\mu) = \{f \in L_\mu^2 : x_1 f(x_1, x_2) \in L_\mu^2\}, \quad (47)$$

$$B_\mu f(x_1, x_2) = x_2 f(x_1, x_2), \quad D(B_\mu) = L_\mu^2. \quad (48)$$

The operator  $A_\mu$  is self-adjoint and the operator  $B_\mu$  is self-adjoint and bounded. These operators commute and therefore the spectral measure  $E_\mu$  of  $A_\mu$  and the spectral measure  $F_\mu$  of  $B_\mu$  commute, as well.

Let  $p(x_1, x_2)$  be a polynomial of the form (1) and  $q(x_1, x_2)$  be a polynomial of the form (1) with  $\beta_{m,n} \in \mathbb{C}$  instead of  $\alpha_{m,n}$ . Then

$$\begin{aligned} (p, q)_\mu &= \sum_{m,n,k,l \in \mathbb{Z}_+} \alpha_{m,n} \overline{\beta_{k,l}} \int_\Pi x_1^{m+k} x_2^{n+l} d\mu \\ &= \sum_{m,n,k,l \in \mathbb{Z}_+} \alpha_{m,n} \overline{\beta_{k,l}} s_{m+k, n+l}, \end{aligned}$$

On the other hand, we may write

$$\begin{aligned} \left( \sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} x_{m,n}, \sum_{k,l \in \mathbb{Z}_+} \beta_{k,l} x_{k,l} \right)_H &= \sum_{m,n,k,l \in \mathbb{Z}_+} \alpha_{m,n} \overline{\beta_{k,l}} (x_{m,n}, x_{k,l})_H \\ &= \sum_{m,n,k,l \in \mathbb{Z}_+} \alpha_{m,n} \overline{\beta_{k,l}} K((m,n), (k,l)) = \sum_{m,n,k,l \in \mathbb{Z}_+} \alpha_{m,n} \overline{\beta_{k,l}} s_{m+k, n+l}. \end{aligned}$$

Therefore

$$(p, q)_\mu = \left( \sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} x_{m,n}, \sum_{k,l \in \mathbb{Z}_+} \beta_{k,l} x_{k,l} \right)_H. \quad (49)$$

Consider the following operator:

$$V[p] = \sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} x_{m,n}, \quad p = \sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} x_1^m x_2^n. \quad (50)$$

Here by  $[p]$  we mean the class of equivalence in  $L_\mu^2$  defined by  $p$ . If two different polynomials  $p$  and  $q$  belong to the same class of equivalence then by (49) we get

$$\begin{aligned} 0 = \|p-q\|_\mu^2 &= (p-q, p-q)_\mu = \left( \sum_{m,n \in \mathbb{Z}_+} (\alpha_{m,n} - \beta_{m,n}) x_{m,n}, \sum_{k,l \in \mathbb{Z}_+} (\alpha_{k,l} - \beta_{k,l}) x_{k,l} \right)_H \\ &= \left\| \sum_{m,n \in \mathbb{Z}_+} \alpha_{m,n} x_{m,n} - \sum_{m,n \in \mathbb{Z}_+} \beta_{m,n} x_{m,n} \right\|_H^2. \end{aligned}$$

Thus, the definition of  $V$  is correct. The operator  $V$  maps the set of all polynomials  $P_{0,\mu}^2$  in  $L_\mu^2$  on  $L$ . By continuity we extend  $V$  to an isometric transformation from the closure of polynomials  $P_\mu^2 = \overline{P_{0,\mu}^2}$  onto  $H$ .

Set  $H_0 := L_\mu^2 \ominus P_\mu^2$ . Introduce the following operator:

$$U := V \oplus E_{H_0}, \quad (51)$$

which maps isometrically  $L_\mu^2$  onto  $\tilde{H} := H \oplus H_0$ . Set

$$\tilde{A} := UA_\mu U^{-1}, \quad \tilde{B} := UB_\mu U^{-1}. \quad (52)$$

Notice that

$$\tilde{A}x_{m,n} = UA_\mu U^{-1}x_{m,n} = UA_\mu x_1^m x_2^n = Ux_1^{m+1} x_2^n = x_{m+1,n},$$

$$\tilde{B}x_{m,n} = UB_\mu U^{-1}x_{m,n} = UB_\mu x_1^m x_2^n = Ux_1^m x_2^{n+1} = x_{m,n+1}.$$

Therefore  $\tilde{A} \supseteq A$  and  $\tilde{B} \supseteq B$ . Let

$$\tilde{A} = \int_{\mathbb{R}} x_1 d\tilde{E}(x_1), \quad \tilde{B} = \int_{[-R,R]} x_2 d\tilde{F}(x_2), \quad (53)$$

where  $\tilde{E}$  and  $\tilde{F}$  are the spectral measures of  $\tilde{A}$  and  $\tilde{B}$ , respectively. Repeating arguments after relation (39) we obtain that

$$x_{m,n} = \tilde{A}^m \tilde{B}^n x_{0,0}, \quad m, n \in \mathbb{Z}_+, \quad (54)$$

$$s_{m,n} = \int_{\Pi} x_1^m x_2^n d((\tilde{E} \times \tilde{F})(x_1, x_2)x_{0,0}, x_{0,0})_{\tilde{H}}, \quad m, n \in \mathbb{Z}_+, \quad (55)$$

where  $(\tilde{E} \times \tilde{F})$  is the product measure of  $\tilde{E}$  and  $\tilde{F}$ . Thus, the measure  $\tilde{\mu} := ((\tilde{E} \times \tilde{F})x_{0,0}, x_{0,0})_{\tilde{H}}$  is a solution of the moment problem. Let  $I_{x_1} \subset \mathbb{R}$ ,  $I_{x_2} \subseteq [-R, R]$  be arbitrary intervals. Observe that

$$\begin{aligned} P_H^{\tilde{H}} \tilde{E}(I_{x_1}) \tilde{F}(I_{x_2}) P_H^{\tilde{H}} &= P_H^{\tilde{H}} \tilde{E}(I_{x_1}) P_H^{\tilde{H}} \tilde{F}(I_{x_2}) P_H^{\tilde{H}} = \mathbf{E}(I_{x_1}) F(I_{x_2}); \\ P_H^{\tilde{H}} \tilde{E}(I_{x_1}) \tilde{F}(I_{x_2}) P_H^{\tilde{H}} &= P_H^{\tilde{H}} \tilde{F}(I_{x_2}) \tilde{E}(I_{x_1}) P_H^{\tilde{H}} = P_H^{\tilde{H}} \tilde{F}(I_{x_2}) P_H^{\tilde{H}} \tilde{E}(I_{x_1}) P_H^{\tilde{H}} \\ &= F(I_{x_2}) \mathbf{E}(I_{x_1}), \end{aligned}$$

and therefore

$$\mathbf{E}(I_{x_1}) F(I_{x_2}) = F(I_{x_2}) \mathbf{E}(I_{x_1}), \quad (56)$$

where  $\mathbf{E}$  is the corresponding spectral function of  $A$  and  $F$  is the spectral function of  $B$ . Then

$$\begin{aligned} \tilde{\mu}(I_{x_1} \times I_{x_2}) &= ((\tilde{E} \times \tilde{F})(I_{x_1} \times I_{x_2})x_{0,0}, x_{0,0})_{\tilde{H}} \\ &= (\tilde{E}(I_{x_1}) \tilde{F}(I_{x_2})x_{0,0}, x_{0,0})_{\tilde{H}} = (P_H^{\tilde{H}} \tilde{F}(I_{x_2}) \tilde{E}(I_{x_1})x_{0,0}, x_{0,0})_{\tilde{H}} \\ &= (P_H^{\tilde{H}} \tilde{F}(I_{x_2}) P_H^{\tilde{H}} \tilde{E}(I_{x_1})x_{0,0}, x_{0,0})_{\tilde{H}} = (F(I_{x_2}) \mathbf{E}(I_{x_1})x_{0,0}, x_{0,0})_H \\ &= (\mathbf{E}(I_{x_1}) F(I_{x_2})x_{0,0}, x_{0,0})_H. \end{aligned}$$

where  $\mathbf{E}$  is the corresponding spectral function of  $A$  and  $F$  is the spectral function of  $B$ . Thus, the measure  $\tilde{\mu}$  admits the representation (45) since the Lebesgue continuation is unique.

Let us show that  $\tilde{\mu} = \mu$ . Consider the following transformation:

$$S : (x_1, x_2) \in \Pi \mapsto \left( \operatorname{Arg} \frac{x_1 - i}{x_1 + i}, x_2 \right) \in \Pi_0, \quad (57)$$

where  $\Pi_0 = [-\pi, \pi) \times [-R, R]$  and  $\operatorname{Arg} e^{iy} = y \in [-\pi, \pi)$ . By virtue of  $V$  we define the following measures:

$$\mu_0(VG) := \mu(G), \quad \tilde{\mu}_0(VG) := \tilde{\mu}(G), \quad G \in \mathfrak{B}(\Pi), \quad (58)$$

It is not hard to see that  $\mu_0$  and  $\tilde{\mu}_0$  are non-negative measures on  $\mathfrak{B}(\Pi_0)$ . Then

$$\int_{\Pi} \left( \frac{x_1 - i}{x_1 + i} \right)^m x_2^n d\mu = \int_{\Pi_0} e^{im\psi} x_2^n d\mu_0, \quad (59)$$

$$\int_{\Pi} \left( \frac{x_1 - i}{x_1 + i} \right)^m x_2^n d\tilde{\mu} = \int_{\Pi_0} e^{im\psi} x_2^n d\tilde{\mu}_0, \quad m \in \mathbb{Z}, n \in \mathbb{Z}_+; \quad (60)$$

and

$$\begin{aligned} \int_{\Pi} \left( \frac{x_1 - i}{x_1 + i} \right)^m x_2^n d\tilde{\mu} &= \int_{\Pi} \left( \frac{x_1 - i}{x_1 + i} \right)^m x_2^n d((\tilde{E} \times \tilde{F})x_{0,0}, x_{0,0})_{\tilde{H}} \\ &= \left( \int_{\Pi} \left( \frac{x_1 - i}{x_1 + i} \right)^m x_2^n d(\tilde{E} \times \tilde{F})x_{0,0}, x_{0,0} \right)_{\tilde{H}} \\ &= \left( \int_{\mathbb{R}} \left( \frac{x_1 - i}{x_1 + i} \right)^m d\tilde{E} \int_{[-R,R]} x_2^n d\tilde{F}x_{0,0}, x_{0,0} \right)_{\tilde{H}} \\ &= \left( \left( (\tilde{A} - iE_{\tilde{H}})(\tilde{A} + iE_{\tilde{H}})^{-1} \right)^m \tilde{B}^n x_{0,0}, x_{0,0} \right)_{\tilde{H}} \\ &= \left( U^{-1} \left( (\tilde{A} - iE_{\tilde{H}})(\tilde{A} + iE_{\tilde{H}})^{-1} \right)^m \tilde{B}^n U1, U1 \right)_{\mu} \\ &= \left( \left( (A_{\mu} - iE_{L_{\mu}^2})(A_{\mu} + iE_{L_{\mu}^2})^{-1} \right)^m B_{\mu}^n 1, 1 \right)_{\mu} \\ &= \int_{\Pi} \left( \frac{x_1 - i}{x_1 + i} \right)^m x_2^n d\mu, \quad m \in \mathbb{Z}, n \in \mathbb{Z}_+. \end{aligned} \quad (61)$$

By virtue of relations (59),(60) and (61) we get

$$\int_{\Pi_0} e^{im\psi} x_2^n d\mu_0 = \int_{\Pi_0} e^{im\psi} x_2^n d\tilde{\mu}_0, \quad m \in \mathbb{Z}, n \in \mathbb{Z}_+. \quad (62)$$

By the Weierstrass theorem we can approximate any continuous function by exponentials and therefore

$$\int_{\Pi_0} f(\psi) x_2^n d\mu_0 = \int_{\Pi_0} f(\psi) x_2^n d\tilde{\mu}_0, \quad n \in \mathbb{Z}_+, \quad (63)$$

for arbitrary continuous functions on  $\Pi_0$ . In particular, we have

$$\int_{\Pi_0} x_1^m x_2^n d\mu_0 = \int_{\Pi_0} x_1^m x_2^n d\tilde{\mu}_0, \quad n, m \in \mathbb{Z}_+. \quad (64)$$

However, the two-dimensional Hausdorff moment problem is determinate ([1]) and therefore we get  $\mu_0 = \tilde{\mu}_0$  and  $\mu = \tilde{\mu}$ . Thus, we have proved that an arbitrary solution  $\mu$  of the moment problem (1) can be represented in the form (45).

Let us check the second assertion of the Theorem. For an arbitrary spectral measure  $\mathbf{E}$  of  $A$  which commutes with the spectral measure  $F$  of  $B$ , by relation (45) we define a non-negative Borel measure  $\mu$  on  $\Pi$ . Let us show that the measure  $\mu$  is a solution of the moment problem (1). Let  $\hat{A}$  be a self-adjoint extension of the operator  $A$  in a Hilbert space  $\hat{H} \supseteq H$ , such that

$$\mathbf{E} = P_H^{\hat{H}} \hat{E},$$

where  $\hat{E}$  is the spectral measure of  $\hat{A}$ . By (40) we get

$$\begin{aligned} x_{m,n} &= A^m B^n x_{0,0} = \hat{A}^m B^n x_{0,0} = P_H^{\hat{H}} \hat{A}^m B^n x_{0,0} \\ &= P_H^{\hat{H}} \lim_{a \rightarrow +\infty} \int_{[-a,a)} x_1^m d\hat{E}(x_1) B^n x_{0,0} = \lim_{a \rightarrow +\infty} P_H^{\hat{H}} \int_{[-a,a)} x_1^m d\hat{E}(x_1) \\ &\quad * B^n x_{0,0} = \lim_{a \rightarrow +\infty} \int_{[-a,a)} x_1^m d\mathbf{E}(x_1) B^n x_{0,0}, \\ &\quad m, n \in \mathbb{Z}_+, \end{aligned} \tag{65}$$

where the integrals are understood as strong limits of the Stieltjes operator sums. We choose arbitrary points

$$\begin{aligned} -a &= x_{1,0} < x_{1,1} < \dots < x_{1,N} = a; \\ \max_{1 \leq i \leq N} |x_{1,i} - x_{1,i-1}| &=: d, \quad N \in \mathbb{N}; \end{aligned} \tag{66}$$

$$\begin{aligned} -R &= x_{2,0} < x_{2,1} < \dots < x_{2,M} = R; \\ \max_{1 \leq j \leq M} |x_{2,j} - x_{2,j-1}| &=: r; \quad M \in \mathbb{N}. \end{aligned} \tag{67}$$

Set  $I_{2,j} = [x_{2,j-1}, x_{2,j})$ , if  $1 \leq j < M$ , and  $I_{2,M} = [x_{2,M-1}, x_{2,M}]$ . Then

$$\begin{aligned} C_a &:= \int_{[-a,a)} x_1^m d\mathbf{E} \int_{[-R,R]} x_2^n dF = \lim_{d \rightarrow 0} \sum_{i=1}^N x_{1,i-1}^m \mathbf{E}([x_{1,i-1}, x_{1,i})) \\ &\quad * \lim_{r \rightarrow 0} \sum_{j=1}^M x_{2,j-1}^n F(I_{2,j}), \end{aligned}$$

where the integral sums converge in the strong operator topology. Then

$$C_a = \lim_{d \rightarrow 0} \lim_{r \rightarrow 0} \sum_{i=1}^N x_{1,i-1}^m \mathbf{E}([x_{1,i-1}, x_{1,i})) \sum_{j=1}^M x_{2,j-1}^n F(I_{2,j})$$

$$= \lim_{d \rightarrow 0} \lim_{r \rightarrow 0} \sum_{i=1}^N \sum_{j=1}^M x_{1,i-1}^m x_{2,j-1}^n \mathbf{E}([x_{1,i-1}, x_{1,i}]) F(I_{2,j}),$$

where the limits are understood in the strong operator topology. Then

$$\begin{aligned} (C_a x_{0,0}, x_{0,0})_H &= \left( \lim_{d \rightarrow 0} \lim_{r \rightarrow 0} \sum_{i=1}^N \sum_{j=1}^M x_{1,i-1}^m x_{2,j-1}^n \mathbf{E}([x_{1,i-1}, x_{1,i}]) F(I_{2,j}) x_{0,0}, x_{0,0} \right)_H \\ &= \lim_{d \rightarrow 0} \lim_{r \rightarrow 0} \sum_{i=1}^N \sum_{j=1}^M x_{1,i-1}^m x_{2,j-1}^n (\mathbf{E}([x_{1,i-1}, x_{1,i}]) F(I_{2,j}) x_{0,0}, x_{0,0})_H \\ &= \lim_{d \rightarrow 0} \lim_{r \rightarrow 0} \sum_{i=1}^N \sum_{j=1}^M x_{1,i-1}^m x_{2,j-1}^n ((\mathbf{E} \times F)([x_{1,i-1}, x_{1,i}] \times I_{2,j}) x_{0,0}, x_{0,0})_H \\ &= \lim_{d \rightarrow 0} \lim_{r \rightarrow 0} \sum_{i=1}^N \sum_{j=1}^M x_{1,i-1}^m x_{2,j-1}^n (\mu([x_{1,i-1}, x_{1,i}] \times I_{2,j}) x_{0,0}, x_{0,0})_H. \end{aligned}$$

Therefore

$$(C_a x_{0,0}, x_{0,0})_H = \lim_{d \rightarrow 0} \lim_{r \rightarrow 0} \int_{[-a,a) \times [-R,R]} f_{d,r}(x_1, x_2) d\mu,$$

where  $f_{d,r}$  is equal to  $x_{1,i-1}^m x_{2,j-1}^n$  on the rectangular  $[x_{1,i-1}, x_{1,i}] \times I_{2,j}$ ,  $1 \leq i \leq N$ ,  $1 \leq j \leq M$ .

If  $r \rightarrow 0$ , then the function  $f_{d,r}(x_1, x_2)$  converges pointwise to a function  $f_d(x_1, x_2)$  which is equal to  $x_{1,i-1}^m x_2^n$  on the rectangular  $[x_{1,i-1}, x_{1,i}] \times [-R, R]$ ,  $1 \leq i \leq N$ . Moreover, the function  $f_{d,r}(x_1, x_2)$  is uniformly bounded. By the Lebesgue we obtain

$$(C_a x_{0,0}, x_{0,0})_H = \lim_{d \rightarrow 0} \int_{[-a,a) \times [-R,R]} f_d(x_1, x_2) d\mu.$$

If  $d \rightarrow 0$ , then the function  $f_d$  converges pointwise to a function  $x_1^m x_2^n$ . Since  $|f_d| \leq a^m R^n$ , by the Lebesgue theorem we get

$$(C_a x_{0,0}, x_{0,0})_H = \int_{[-a,a) \times [-R,R]} x_1^m x_2^n d\mu. \quad (68)$$

By virtue of relations (65) and (68) we get

$$s_{m,n} = (x_{m,n}, x_{0,0})_H = \lim_{a \rightarrow +\infty} (C_a x_{0,0}, x_{0,0})_H$$



$$= \lim_{a \rightarrow +\infty} \int_{[-a,a) \times [-R,R]} x_1^m x_2^n d\mu = \int_{\Pi} x_1^m x_2^n d\mu. \quad (69)$$

Thus, the measure  $\mu$  is a solution of the moment problem (1).

Let us prove the last assertion of the Theorem. Suppose to the contrary that two different spectral measures  $\mathbf{E}_1$  and  $\mathbf{E}_2$  of  $A$  commute with the spectral measure  $F$  of  $B$  and produce by relation (45) the same solution  $\mu$  of the moment problem. Choose an arbitrary  $z \in \mathbb{C} \setminus \mathbb{R}$ . Then

$$\begin{aligned} \int_{\Pi} \frac{x_1^m}{x_1 - z} x_2^n d\mu &= \int_{\Pi} \frac{x_1^m}{x_1 - z} x_2^n ((\mathbf{E}_k \times F)(\delta) x_{0,0}, x_{0,0})_H \\ &= \lim_{a \rightarrow +\infty} \int_{[-a,a) \times [-R,R]} \frac{x_1^m}{x_1 - z} x_2^n d((\mathbf{E}_k \times F)(\delta) x_{0,0}, x_{0,0})_H, \quad k = 1, 2. \end{aligned} \quad (70)$$

Consider arbitrary partitions of the type (66),(67). Then

$$\begin{aligned} D_a &:= \int_{[-a,a) \times [-R,R]} \frac{x_1^m}{x_1 - z} x_2^n d((\mathbf{E}_k \times F)(\delta) x_{0,0}, x_{0,0})_H \\ &= \lim_{d \rightarrow 0} \lim_{r \rightarrow 0} \int_{[-a,a) \times [-R,R]} g_{z;d,r}(x, \varphi) d((\mathbf{E}_k \times F)(\delta) x_{0,0}, x_{0,0})_H. \end{aligned}$$

Here the function  $g_{z;d,r}(x_1, x_2)$  is equal to  $\frac{x_{1,i-1}^m}{x_{1,i-1} - z} x_{2,j-1}^n$  on the rectangular  $[x_{i-1}, x_i) \times I_{2,j-1}$ ,  $1 \leq i \leq N$ ,  $1 \leq j \leq M$ . Then

$$\begin{aligned} D_a &= \lim_{d \rightarrow 0} \lim_{r \rightarrow 0} \sum_{i=1}^N \sum_{j=1}^M \frac{x_{1,i-1}^m}{x_{1,i-1} - z} x_{2,j-1}^n (\mathbf{E}_k([x_{1,i-1}, x_{1,i})) F(I_{2,j}) x_{0,0}, x_{0,0})_H \\ &= \lim_{d \rightarrow 0} \lim_{r \rightarrow 0} \left( \sum_{i=1}^N \frac{x_{1,i-1}^m}{x_{1,i-1} - z} \mathbf{E}_k([x_{i-1}, x_i)) \sum_{j=1}^M x_{2,j-1}^n F(I_{2,j}) x_{0,0}, x_{0,0} \right)_H \\ &= \left( \int_{[-a,a)} \frac{x_1^m}{x_1 - z} d\mathbf{E}_k \int_{[-R,R]} x_2^n dF x_{0,0}, x_{0,0} \right)_H. \end{aligned}$$

Let  $n = n_1 + n_2$ ,  $n_1, n_2 \in \mathbb{Z}_+$ . Then we may write:

$$\begin{aligned} D_a &= \left( B^{n_1} \int_{[-a,a)} \frac{x_1^m}{x_1 - z} d\mathbf{E}_k B^{n_2} x_{0,0}, x_{0,0} \right)_H \\ &= \left( \int_{[-a,a)} \frac{x_1^m}{x_1 - z} d\mathbf{E}_k x_{0,n_2}, x_{0,n_1} \right)_H. \end{aligned}$$

By (70) we get

$$\begin{aligned}
\int_{\Pi} \frac{x_1^m}{x_1 - z} x_2^n d\mu &= \lim_{a \rightarrow +\infty} D_a = \lim_{a \rightarrow +\infty} \left( \int_{[-a, a)} \frac{x_1^m}{x_1 - z} d\widehat{E}_k x_{0, n_2}, x_{0, n_1} \right)_{\widehat{H}_k} \\
&= \left( \int_{\mathbb{R}} \frac{x_1^m}{x_1 - z} d\widehat{E}_k x_{0, n_2}, x_{0, n_1} \right)_{\widehat{H}_k} = \left( \widehat{A}_k^{m_2} R_z(\widehat{A}_k) \widehat{A}_k^{m_1} x_{0, n_2}, x_{0, n_1} \right)_{\widehat{H}_k} \\
&= \left( R_z(\widehat{A}_k) x_{m_1, n_2}, x_{m_2, n_1} \right)_H, \tag{71}
\end{aligned}$$

where  $m_1, m_2 \in \mathbb{Z}_+ : m_1 + m_2 = m$ , and  $\widehat{A}_k$  is a self-adjoint extension of  $A$  in a Hilbert space  $\widehat{H}_k \supseteq H$  such that its spectral measure  $\widehat{E}_k$  generates  $\mathbf{E}_k$ :  $\mathbf{E}_k = P_H^{\widehat{H}_k} \widehat{E}_k$ ;  $k = 1, 2$ .

Relation (71) shows that the generalized resolvents corresponding to  $\mathbf{E}_k$ ,  $k = 1, 2$ , coincide. This means that the spectral measures  $\mathbf{E}_1$  and  $\mathbf{E}_2$  coincide. We obtained a contradiction. This completes the proof.  $\square$

**Definition 3.1** *A solution  $\mu$  of the moment problem (1) is said to be **canonical** if it is generated by relation (45) where  $\mathbf{E}$  is an **orthogonal** spectral measure of  $A$  which commutes with the spectral measure of  $B$ . Orthogonal spectral measures are those measures which are the spectral measures of self-adjoint extensions of  $A$  inside  $H$ .*

Let the moment problem (1) be given and conditions (3),(4) hold. Let us describe canonical solutions of the two-dimensional moment problem in a strip. In the proof of Theorem 2.2 we have constructed one canonical solution, see relation (41). Let  $\mu$  be an arbitrary canonical solution and  $\mathbf{E}$  be the corresponding orthogonal spectral measure of  $A$ . Let  $\widetilde{A}$  be the self-adjoint operator in  $H$  which corresponds to  $\mathbf{E}$ . Consider the Cayley transformation of  $\widetilde{A}$ :

$$U_{\widetilde{A}} = (\widetilde{A} + iE_H)(\widetilde{A} - iE_H)^{-1} \supseteq V_A, \tag{72}$$

where  $V_A$  is defined by (20). Since  $\mathbf{E}$  commutes with the spectral measure  $F$  of  $B$ , then  $U_{\widetilde{A}}$  commutes with  $B$  and with  $U_B$ . By relation (31) the operator  $U_{\widetilde{A}}$  have the following form:

$$U_{\widetilde{A}} = V_A \oplus \widetilde{U}_{2,4}, \tag{73}$$

where  $\widetilde{U}_{2,4}$  is an isometric operator which maps  $H_2$  onto  $H_4$ , and commutes with  $U_B$ . Let the operator  $U_{2,4}$  be defined by (35). Then the following operator

$$U_2 = U_{2,4}^{-1} \widetilde{U}_{2,4}, \tag{74}$$

is a unitary operator in  $H_2$  which commutes with  $U_{B;H_2}$ .

Denote by  $\mathbf{S}(U_B; H_2)$  a set of all unitary operators in  $H_2$  which commute with  $U_{B;H_2}$ . Choose an arbitrary operator  $\widehat{U}_2 \in \mathbf{S}(U_B; H_2)$ . Define  $\widehat{U}_{2,4}$  by the following relation:

$$\widehat{U}_{2,4} = U_{2,4}\widehat{U}_2. \quad (75)$$

Notice that  $\widehat{U}_{2,4}U_B h = \widehat{U}_{2,4}U_B h$ ,  $h \in H_2$ . Then we define a unitary operator  $U = V_A \oplus \widehat{U}_{2,4}$  and its Cayley transformation  $\widehat{A}$  which commute with the operator  $B$ . Repeating arguments before (41) we get a canonical solution of the moment problem.

Thus, all canonical solutions of the Devinatz moment problem are generated by operators  $\widehat{U}_2 \in \mathbf{S}(U_B; H_2)$ . Notice that different operators  $U', U'' \in \mathbf{S}(U_B; H_2)$  produce different orthogonal spectral measures  $\mathbf{E}', \mathbf{E}$ . By Theorem 3.1, these spectral measures produce different solutions of the moment problem.

Recall some definitions from [14]. A pair  $(Y, \mathfrak{A})$ , where  $Y$  is an arbitrary set and  $\mathfrak{A}$  is a fixed  $\sigma$ -algebra of subsets of  $A$  is said to be a *measurable space*. A triple  $(Y, \mathfrak{A}, \mu)$ , where  $(Y, \mathfrak{A})$  is a measurable space and  $\mu$  is a measure on  $\mathfrak{A}$  is said to be a *space with a measure*.

Let  $(Y, \mathfrak{A})$  be a measurable space,  $\mathbf{H}$  be a Hilbert space and  $\mathcal{P} = \mathcal{P}(\mathbf{H})$  be a set of all orthogonal projectors in  $\mathbf{H}$ . A countably additive mapping  $E : \mathfrak{A} \rightarrow \mathcal{P}$ ,  $E(Y) = E_{\mathbf{H}}$ , is said to be a *spectral measure* in  $\mathbf{H}$ . A set  $(Y, \mathfrak{A}, H, E)$  is said to be a *space with a spectral measure*. By  $S(Y, E)$  one means a set of all  $E$ -measurable  $E$ -a.e. finite complex-valued functions on  $Y$ .

Let  $(Y, \mathfrak{A}, \mu)$  be a separable space with a  $\sigma$ -finite measure and to  $\mu$ -everyone  $y \in Y$  it corresponds a Hilbert space  $G(y)$ . A function  $N(y) = \dim G(y)$  is called the *dimension function*. It is supposed to be  $\mu$ -measurable. Let  $\Omega$  be a set of vector-valued functions  $g(y)$  with values in  $G(y)$  which are defined  $\mu$ -everywhere and are measurable with respect to some base of measurability. A set of (classes of equivalence) of such functions with the finite norm

$$\|g\|_{\mathcal{H}}^2 = \int |g(y)|_{G(y)}^2 d\mu(y) < \infty \quad (76)$$

form a Hilbert space  $\mathcal{H}$  with the scalar product given by

$$(g_1, g_2)_{\mathcal{H}} = \int (g_1, g_2)_{G(y)} d\mu(y). \quad (77)$$

The space  $\mathcal{H} = \mathcal{H}_{\mu,N} = \int_Y \oplus G(y) d\mu(y)$  is said to be a *direct integral of Hilbert spaces*. Consider the following operator

$$\mathbf{X}(\delta)g = \chi_\delta g, \quad g \in \mathcal{H}, \quad \delta \in \mathfrak{A}, \quad (78)$$

where  $\chi_\delta$  is the characteristic function of the set  $\delta$ . The operator  $\mathbf{X}$  is a spectral measure in  $\mathcal{H}$ .

Let  $t(y)$  be a measurable operator-valued function with values in  $\mathbf{B}(G(y))$  which is  $\mu$ -a.e. defined and  $\mu - \sup \|t(y)\|_{G(y)} < \infty$ . The operator

$$T : g(y) \mapsto t(y)g(y), \quad (79)$$

is said to be *decomposable*. It is a bounded operator in  $\mathcal{H}$  which commutes with  $\mathbf{X}(\delta)$ ,  $\forall \delta \in \mathfrak{A}$ . Moreover, every bounded operator in  $\mathcal{H}$  which commutes with  $\mathbf{X}(\delta)$ ,  $\forall \delta \in \mathfrak{A}$ , is decomposable [14]. In the case  $t(y) = \varphi(y)E_{G(y)}$ , where  $\varphi \in S(Y, \mu)$ , we set  $T =: Q_\varphi$ . The decomposable operator is unitary if and only if  $\mu$ -a.e. the operator  $t(y)$  is unitary.

Return to the investigation of canonical solutions. Consider the spectral measure  $F_2$  of the operator  $U_{B;H_2}$  in  $H_2$ . There exists an element  $h \in H_2$  of the maximal type, i.e. the non-negative Borel measure

$$\mu(\delta) := (F_2(\delta)h, h), \quad \delta \in \mathfrak{B}([-\pi, \pi]), \quad (80)$$

has the maximal type between all such measures (generated by other elements of  $H_2$ ). This type is said to be the *spectral type* of the measure  $F_2$ . Let  $N_2$  be the multiplicity function of the measure  $F_2$ . Then there exists a unitary transformation  $W$  of the space  $H_2$  on  $\mathcal{H} = \mathcal{H}_{\mu,N_2}$  such that

$$WU_{B;H_2}W^{-1} = Q_{e^{iy}}, \quad WF_2(\delta)W^{-1} = \mathbf{X}(\delta). \quad (81)$$

Notice that  $\widehat{U}_2 \in \mathbf{S}(U_B; H_2)$  if and only if the operator

$$V_2 := W\widehat{U}_2W^{-1}, \quad (82)$$

is unitary and commutes with  $\mathbf{X}(\delta)$ ,  $\forall \delta \in [-\pi, \pi]$ . The latter is equivalent to the condition that  $V_2$  is decomposable and the values of the corresponding operator-valued function  $t(y)$  are  $\mu$ -a.e. unitary operators. A set of all decomposable operators in  $\mathcal{H}$  such that the values of the corresponding operator-valued function  $t(y)$  are  $\mu$ -a.e. unitary operators we denote by  $\mathbf{D}(U_B; H_2)$ .

**Theorem 3.2** *Let the moment problem (1) be given. In the conditions of Theorem 3.1 all canonical solutions of the moment problem have the form (45) where the spectral measures  $\mathbf{E}$  of the operator  $A$  are constructed by operators from  $\mathbf{D}(U_B; H_2)$ . Namely, for an arbitrary  $V_2 \in \mathbf{D}(U_B; H_2)$  we set  $U_2 = W^{-1}V_2W$ ,  $\widehat{U}_{2,4} = U_{2,4}\widehat{U}_2$ ,  $U = V_A \oplus \widehat{U}_{2,4}$ ,  $\widehat{A} = i(U + E_H)(U - E_H)^{-1}$ , and then  $\mathbf{E}$  is the spectral measure of  $\widehat{A}$ .*

*Moreover, the correspondence between  $\mathbf{D}(U_B; H_2)$  and a set of all canonical solutions of the moment problem is bijective.*

**Proof.** The proof follows from the previous considerations.  $\square$

Consider the moment problem (1) and suppose that conditions (3),(4) hold. Let us turn to a parameterization of all solutions of the moment problem. We shall use Theorem 3.1. Consider relation (45). The spectral measure  $\mathbf{E}$  commutes with the operator  $U_B$ . Choose an arbitrary  $z \in \mathbb{C} \setminus \mathbb{R}$ . By virtue of relation (44) we may write:

$$\begin{aligned} (U_B \mathbf{R}_z(A)x, y)_H &= (\mathbf{R}_z(A)x, U_B^* y)_H = \int_{\mathbb{R}} \frac{1}{t-z} d(\mathbf{E}(t)x, U_B^* y)_H \\ &= \int_{\mathbb{R}} \frac{1}{t-z} d(U_B \mathbf{E}(t)x, y)_H = \int_{\mathbb{R}} \frac{1}{t-z} d(\mathbf{E}(t)U_B x, y)_H, \quad x, y \in H; \end{aligned} \quad (83)$$

$$(\mathbf{R}_z(A)U_B x, y)_H = \int_{\mathbb{R}} \frac{1}{t-z} d(\mathbf{E}(t)U_B x, y)_H, \quad x, y \in H, \quad (84)$$

where  $\mathbf{R}_z(A)$  is the generalized resolvent which corresponds to  $\mathbf{E}$ . Therefore we get

$$\mathbf{R}_z(A)U_B = U_B \mathbf{R}_z(A), \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (85)$$

On the other hand, if relation (85) holds, then

$$\int_{\mathbb{R}} \frac{1}{t-z} d(\mathbf{E}U_B x, y)_H = \int_{\mathbb{R}} \frac{1}{t-z} d(U_B \mathbf{E}x, y)_H, \quad x, y \in H, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (86)$$

By the Stieltjes inversion formula [1], we obtain that  $\mathbf{E}$  commutes with  $U_B$ . We denote by  $\mathbf{M}(A, B)$  a set of all generalized resolvents  $\mathbf{R}_z(A)$  of  $A$  which satisfy relation (85).

Recall some known facts from [9] which we shall need here. Let  $K$  be a closed symmetric operator in a Hilbert space  $\mathbf{H}$ , with the domain  $D(K)$ ,  $\overline{D(K)} = \mathbf{H}$ . Set  $N_\lambda = N_\lambda(K) = \mathbf{H} \ominus \Delta_K(\lambda)$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ .

Consider an arbitrary bounded linear operator  $C$ , which maps  $N_i$  into  $N_{-i}$ . For

$$g = f + C\psi - \psi, \quad f \in D(K), \quad \psi \in N_i, \quad (87)$$

we set

$$K_C g = Kf + iC\psi + i\psi. \quad (88)$$

The operator  $K_C$  is said to be a *quasiself-adjoint extension* of the operator  $K$ , defined by the operator  $K$ .

The following theorem can be found in [9, Theorem 7]:

**Theorem 3.3** *Let  $K$  be a closed symmetric operator in a Hilbert space  $\mathbf{H}$  with the domain  $D(K)$ ,  $\overline{D(K)} = \mathbf{H}$ . All generalized resolvents of the operator  $K$  have the following form:*

$$\mathbf{R}_\lambda(K) = \begin{cases} (K_{F(\lambda)} - \lambda E_{\mathbf{H}})^{-1}, & \text{Im } \lambda > 0 \\ (K_{F^*(\bar{\lambda})} - \lambda E_{\mathbf{H}})^{-1}, & \text{Im } \lambda < 0 \end{cases}, \quad (89)$$

where  $F(\lambda)$  is an analytic in  $\mathbb{C}_+$  operator-valued function, which values are contractions which map  $N_i(A) = H_2$  into  $N_{-i}(A) = H_4$  ( $\|F(\lambda)\| \leq 1$ ), and  $K_{F(\lambda)}$  is the quasiself-adjoint extension of  $K$  defined by  $F(\lambda)$ .

On the other hand, for any operator function  $F(\lambda)$  having the above properties there corresponds by relation (89) a generalized resolvent of  $K$ .

Observe that the correspondence between all generalized resolvents and functions  $F(\lambda)$  in Theorem 3.3 is bijective [9].

Return to the study of the moment problem (1). Let us describe the set  $\mathbf{M}(A, B)$ . Choose an arbitrary  $\mathbf{R}_\lambda \in \mathbf{M}(A, B)$ . By (89) we get

$$\mathbf{R}_\lambda = (A_{F(\lambda)} - \lambda E_H)^{-1}, \quad \text{Im } \lambda > 0, \quad (90)$$

where  $F(\lambda)$  is an analytic in  $\mathbb{C}_+$  operator-valued function, which values are contractions which map  $H_2$  into  $H_4$ , and  $A_{F(\lambda)}$  is the quasiself-adjoint extension of  $A$  defined by  $F(\lambda)$ . Then

$$A_{F(\lambda)} = \mathbf{R}_\lambda^{-1} + \lambda E_H, \quad \text{Im } \lambda > 0.$$

By virtue of relation (85) we obtain

$$U_B A_{F(\lambda)} h = A_{F(\lambda)} U_B h, \quad h \in D(A_{F(\lambda)}), \quad \lambda \in \mathbb{C}_+. \quad (91)$$

Consider the following operators

$$W_\lambda := (A_{F(\lambda)} + iE_H)(A_{F(\lambda)} - iE_H)^{-1} = E_H + 2i(A_{F(\lambda)} - iE_H)^{-1}, \quad (92)$$

$$V_A = (A + iE_H)(A - iE_H)^{-1} = E_H + 2i(A - iE_H)^{-1}, \quad (93)$$

where  $\lambda \in \mathbb{C}_+$ . Notice that ([9])

$$W_\lambda = V_A \oplus F(\lambda), \quad \lambda \in \mathbb{C}_+. \quad (94)$$

The operator  $(A_{F(\lambda)} - iE_H)^{-1}$  is defined on the whole  $H$ , see [9, p.79]. By relation (91) we obtain

$$U_B(A_{F(\lambda)} - iE_H)^{-1}h = (A_{F(\lambda)} - iE_H)^{-1}U_Bh, \quad h \in H, \lambda \in \mathbb{C}_+. \quad (95)$$

Then

$$U_BW_\lambda = W_\lambda U_B, \quad \lambda \in \mathbb{C}_+. \quad (96)$$

Recall that by Proposition 2.1 the operator  $U_B$  reduces the subspaces  $H_j$ ,  $1 \leq j \leq 4$ , and  $U_BV_A = V_AU_B$ . If we choose an arbitrary  $h \in H_2$  and apply relations (96),(94), we get

$$U_BF(\lambda) = F(\lambda)U_B, \quad \lambda \in \mathbb{C}_+. \quad (97)$$

Denote by  $\mathbf{F}(A, B)$  a set of all analytic in  $\mathbb{C}_+$  operator-valued functions which values are contractions which map  $H_2$  into  $H_4$  and which satisfy relation (97). Thus, for an arbitrary  $\mathbf{R}_\lambda \in \mathbf{M}(A, B)$  the corresponding function  $F(\lambda)$  belongs to  $\mathbf{F}(A, B)$ .

On the other hand, choose an arbitrary  $F(\lambda) \in \mathbf{F}(A, B)$ . Then we derive (96) with  $W_\lambda$  defined by (92). Then we get (95),(91) and therefore

$$U_B\mathbf{R}_\lambda = \mathbf{R}_\lambda U_B, \quad \lambda \in \mathbb{C}_+. \quad (98)$$

Calculating the conjugate operators for the both sides of the last equality we conclude that this relation holds for all  $\lambda \in \mathbb{C}$ .

Consider the spectral measure  $F_2$  of the operator  $U_{B;H_2}$  in  $H_2$ . We shall use relation (81). Observe that  $F(\lambda) \in \mathbf{F}(A, B)$  if and only if the operator-valued function

$$G(\lambda) := WU_{2,4}^{-1}F(\lambda)W^{-1}, \quad \lambda \in \mathbb{C}_+, \quad (99)$$

is analytic in  $\mathbb{C}_+$  and has values which are contractions in  $\mathcal{H}$  which commute with  $\mathbf{X}(\delta)$ ,  $\forall \delta \in [-\pi, \pi)$ .

This means that for an arbitrary  $\lambda \in \mathbb{C}_+$  the operator  $G(\lambda)$  is decomposable and the values of the corresponding operator-valued function  $t(y)$  are  $\mu$ -a.e. contractions. A set of all decomposable operators in  $\mathcal{H}$  such that the values of the corresponding operator-valued function  $t(y)$  are  $\mu$ -a.e. contractions we denote by  $\mathbf{T}(B; H_2)$ . A set of all analytic in  $\mathbb{C}_+$  operator-valued functions  $G(\lambda)$  with values in  $\mathbf{T}(B; H_2)$  we denote by  $\mathbf{G}(A, B)$ .

**Theorem 3.4** *Let the two-dimensional moment problem in a strip (1) be given. In the conditions of Theorem 3.1 all solutions of the moment problem have the form (45) where the spectral measures  $\mathbf{E}$  of the operator  $A$  are defined by the corresponding generalized resolvents  $\mathbf{R}_\lambda$  which are constructed by the following relation:*

$$\mathbf{R}_\lambda = (A_{F(\lambda)} - \lambda E_H)^{-1}, \quad \text{Im } \lambda > 0, \quad (100)$$

where  $F(\lambda) = U_{2,4}W^{-1}G(\lambda)W$ ,  $G(\lambda) \in \mathbf{G}(A, B)$ .

Moreover, the correspondence between  $\mathbf{G}(A, B)$  and a set of all solutions of the moment problem is bijective.

**Proof.** The proof follows from the previous considerations.  $\square$

## 4 The complex moment problem in a strip.

In this Section we shall analyze the following problem: to find a non-negative Borel measure  $\sigma$  in a strip

$$\Psi = \Psi(R) = \{z \in \mathbb{C} : |\text{Im } z| \leq R\}, \quad R > 0,$$

such that

$$\int_{\Psi} z^m \bar{z}^n d\sigma = a_{m,n}, \quad m, n \in \mathbb{Z}_+, \quad (101)$$

where  $\{a_{m,n}\}_{m,n \in \mathbb{Z}_+}$  is a prescribed sequence of complex numbers. This problem is said to be **the complex moment problem in a strip**. Of course, the strips  $\Psi$  and  $\Pi$  are the same sets in accordance with the canonical identification of  $\mathbb{C}$  with  $\mathbb{R}^2$ :

$$z = x_1 + x_2 i, \quad x_1 = \text{Re } z, \quad x_2 = \text{Im } z, \quad z \in \mathbb{C}, \quad (x_1, x_2) \in \mathbb{R}^2. \quad (102)$$

Let  $\sigma$  be a solution of the complex moment problem in a strip (101). The measure  $\sigma$ , viewed as a measure in  $\mathbb{R}^2$ , we shall denote by  $\mu_\sigma$ . Then

$$\begin{aligned} s_{m,n} &:= \int_{\Pi} x_1^m x_2^n d\mu_\sigma = \int_{\Psi} \left( \frac{z + \bar{z}}{2} \right)^m \left( \frac{z - \bar{z}}{2i} \right)^n d\sigma \\ &= \frac{1}{2^m (2i)^n} \sum_{k=0}^m \sum_{j=0}^n C_k^m C_j^n (-1)^{n-j} \int_{\Psi} z^{k+j} \bar{z}^{m-k+n-j} d\sigma \\ &= \frac{1}{2^m (2i)^n} \sum_{k=0}^m \sum_{j=0}^n (-1)^{n-j} C_k^m C_j^n a_{k+j, m-k+n-j}, \end{aligned} \quad (103)$$



where  $C_k^n = \frac{n!}{k!(n-k)!}$ . Then

$$\begin{aligned} a_{m,n} &= \int_{\Psi} z^m \bar{z}^n d\sigma = \int_{\Pi} (x_1 + ix_2)^m (x_1 - ix_2)^n d\mu_{\sigma} \\ &= \sum_{r=0}^m \sum_{l=0}^n C_r^m C_l^n (-1)^{n-l} \int_{\Pi} x_1^{r+l} (ix_2)^{m-r+n-l} d\mu_{\sigma} \\ &= \sum_{r=0}^m \sum_{l=0}^n C_r^m C_l^n (-1)^{n-l} i^{m-r+n-l} s_{r+l, m-r+n-l}; \end{aligned}$$

and therefore

$$a_{m,n} = \sum_{r=0}^m \sum_{l=0}^n C_r^m C_l^n (-1)^{n-l} i^{m-r+n-l} s_{r+l, m-r+n-l}, \quad m, n \in \mathbb{Z}_+, \quad (104)$$

where

$$s_{m,n} = \frac{1}{2^{m+n}} \sum_{k=0}^m \sum_{j=0}^n (-1)^j C_k^m C_j^n a_{k+j, m-k+n-j}, \quad m, n \in \mathbb{Z}_+. \quad (105)$$

Since  $\mu_{\sigma}$  is a solution of the two-dimensional moment problem in a strip, then conditions (3),(4) hold.

**Theorem 4.1** *Let the complex moment problem in a strip (101) be given. This problem has a solution if and only if conditions (3),(4) and (104) with  $s_{m,n}$  defined by (105) hold for arbitrary complex numbers  $\alpha_{m,n}$  such that all but finite numbers are zeros.*

**Proof.** It remains to prove the sufficiency. Suppose that for the complex moment problem in a strip (1) conditions (3),(4) and (104) hold. By Theorem 2.2 we obtain that there exists a solution  $\mu$  of the two-dimensional moment problem with moments  $s_{m,n}$  defined by (105). The measure  $\mu$ , viewed as a measure in  $\mathbb{C}$ , we shall denote by  $\sigma_{\mu}$ . Then

$$\begin{aligned} \int_{\Psi} z^m \bar{z}^n d\sigma_{\mu} &= \int_{\Pi} (x_1 + ix_2)^m (x_1 - ix_2)^n d\mu \\ &= \sum_{r=0}^m \sum_{l=0}^n C_r^m C_l^n (-1)^{n-l} \int_{\Pi} x_1^{r+l} (ix_2)^{m-r+n-l} d\mu \\ &= \sum_{r=0}^m \sum_{l=0}^n C_r^m C_l^n (-1)^{n-l} i^{m-r+n-l} s_{r+l, m-r+n-l} = a_{m,n}, \end{aligned}$$

where the last equality follows from (104).  $\square$

**Theorem 4.2** *Let the complex moment problem in a strip (101) be given and conditions (3),(4) and (104) hold for arbitrary complex numbers  $\alpha_{m,n}$  such that all but finite numbers are zeros. All solutions of the moment problem (101) are solutions of the two-dimensional moment problem (1) with  $s_{m,n}$  defined by (105), viewed as measures on  $\mathbb{C}$ . Therefore all solutions of the moment problem (101) have a parameterization provided by Theorem 3.4.*

**Proof.** The proof is straightforward.  $\square$

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### **The two-dimensional moment problem in a strip.**

**S.M. Zagorodnyuk**

In this paper we study the two-dimensional moment problem in a strip  $\Pi(R) = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \leq R\}$ ,  $R > 0$ . We obtained a solvability criterion for this moment problem. We derived a parameterization of all solutions of the moment problem. An abstract operator approach and results of Godič, Lucenko and Shtraus are used.

Key words: moment problem, measure, generalized resolvent.

MSC 2000: 44A60, 30E05.